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ALMOST PERIODIC OPERATORS IN VN(G)

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ABSTRACT. Let G be a locally compact group, $A(G)$ the Fourier algebra of G, $B(G)$ the Fourier-Stieltjes algebra of G and $\mathrm{VN}(G)$ the von Neumann algebra generated by the left regular representation λ of G. Then $A(G)$ is the predual of $\mathrm{VN}(G)$; $\mathrm{VN}(G)$ is a $B(G)$-module and $A(G)$ is a closed ideal of $B(G)$. Let $\mathrm{AP}(\widehat{G}) = \{T \in \mathrm{VN}(G) : u \mapsto u \cdot T$ is a compact operator from $A(G)$ into $\mathrm{VN}(G)\}$, the space of almost periodic operators in $\mathrm{VN}(G)$. Let $C_\delta^*(G)$ be the C^*-algebra generated by $\{\lambda(x) : x \in G\}$. Then $C_\delta^*(G) \subset \mathrm{AP}(\widehat{G})$. For a compact G, let E be the rank one operator on $L^2(G)$ that sends $h \in L^2(G)$ to the constant function $\int h(x)\,dx$. We have the following results: (1) There exists a compact group G such that $E \in \mathrm{AP}(\widehat{G}) \backslash C_\delta^*(G)$. (2) For a compact Lie group G, $E \in \mathrm{AP}(\widehat{G}) \Leftrightarrow E \in C_\delta^*(G) \Leftrightarrow L^\infty(G)$ has a unique left invariant mean $\Leftrightarrow G$ is semisimple. (3) If G is an extension of a locally compact abelian group by an amenable discrete group then $\mathrm{AP}(\widehat{G}) = C_\delta^*(G)$. (4) Let $G = \mathbf{F}_r$, the free group with r generators, $1 < r < \infty$. If $T \in \mathrm{VN}(G)$ and $u \mapsto u \cdot T$ is a compact operator from $B(G)$ into $\mathrm{VN}(G)$ then $T \in C_\delta^*(G)$.

1. INTRODUCTION

Let G be a locally compact group and $\mathrm{VN}(G)$ the von Neumann algebra generated by the left regular representation λ of G. Then the predual of $\mathrm{VN}(G)$ can be realized as an algebra of continuous functions on G, namely the Fourier algebra $A(G)$ of G. Moreover, $\mathrm{VN}(G)$ is an $A(G)$-module where for $u \in A(G)$ and $T \in \mathrm{VN}(G)$, $u \cdot T \in \mathrm{VN}(G)$ is defined by $\langle u \cdot T, v\rangle = \langle T, uv\rangle$, $v \in A(G)$. The algebras $A(G)$ and $\mathrm{VN}(G)$ were defined and studied in detail by P. Eymard [17].

For $T \in \mathrm{VN}(G)$, we will denote the operator that sends u in $A(G)$ to $u \cdot T$ in $\mathrm{VN}(G)$ by T^A. Following Dunkl and Ramirez [16]. $T \in \mathrm{VN}(G)$ is said to be almost periodic (a.p.) if T^A is a compact operator from $A(G)$ into $\mathrm{VN}(G)$ and the space of all almost periodic elements in $\mathrm{VN}(G)$ will be denoted by $\mathrm{AP}(\widehat{G})$. Let $C_\delta^*(G)$ be the C^*-algebra generated by $\{\lambda(x) : x \in G\}$.

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It is known that $C^*_\delta(G) \subset \mathrm{AP}(\widehat{G})$ and $C^*_\delta(G) = \mathrm{AP}(\widehat{G})$ if G is either a locally compact abelian group or a discrete amenable group (see Dunkl and Ramirez [16] and Granirer [20]). In this paper we will study whether $C^*_\delta(G) = \mathrm{AP}(\widehat{G})$ for other locally compact groups G.

Let G be an infinite compact group and $E = E_G$ the rank one operator in $\mathrm{VN}(G)$ given by $E(g) = (\int g(x)\,dx) \cdot 1_G$ where $g \in L^2(G)$ and 1_G denotes the constant one function on G. Dunkl and Ramirez [15] have shown that $E \in \mathrm{AP}(\widehat{G})$ if, for each positive integer n, G has only finitely many inequivalent continuous irreducible unitary rperesentations of degree n. It is also known that $E \in C^*_\delta(G)$ if and only if the trivial representation of G_d (the discrete version of G) is not weakly contained in the left regular representation of G_d on $L^2_0(G) = \{f \in L^2(G) \colon \int f(x)\,dx = 0\}$ (see [7]). Using these two results, we will prove in §3 that there exists an infinite compact group G such that $E \in \mathrm{AP}(\widehat{G})\backslash C^*_\delta(G)$; in particular, $C^*_\delta(G) \subsetneqq \mathrm{AP}(\widehat{G})$. On the other hand, by applying the results in Margulis [33] and Drinfel′d [11], we are able to show that if G is a compact Lie group then the following conditions are equivalent:

(1) $E \in \mathrm{AP}(\widehat{G})$;
(2) $E \in C^*_\delta(G)$;
(3) $L^\infty(G)$ has a unique left invariant mean;
(4) G is semisimple.

The uniqueness and nonuniqueness of left invariant means on compact groups have been studied by Granirer [19], Rudin [43] and Chou, Lau and Rosenblatt [7]. For $G = \mathrm{SO}(n)$, $n \geq 3$, the fact that $E \in C^*_\delta(G)$ is equivalent to a key fact that is needed in the solution of the Banach-Ruziewicz problem in S^{n-1} (see [33, 44, 11, 7]). On the other hand, it is relatively easy to show that $E \in \mathrm{AP}(\widehat{G})$ for such G. It would therefore be interesting if one could decide whether $\mathrm{AP}(\widehat{G}) = C^*_\delta(G)$ if $G = \mathrm{SO}(n)$, $n \geq 3$.

In §4, we will show that if G is an extension of a locally compact abelian group by a discrete amenable group then $\mathrm{AP}(\widehat{G}) = C^*_\delta(G)$. In particular, by a theorem of C. C. Moore [36], $\mathrm{AP}(\widehat{G}) = C^*_\delta(G)$ if G is of bounded representation type. Dunkl and Ramirez have shown in [14] that $\mathrm{AP}(\widehat{G})$ is a C^*-algebra if G is a compact group of bounded representation type.

Let $B(G)$ be the Fourier-Stieltjes algebra of a locally compact group G. Then $A(G)$ is a closed ideal of $B(G)$ and $\mathrm{VN}(G)$ is also a $B(G)$-module (see [17]). $T \in \mathrm{VN}(G)$ is said to be $B(G)$-almost periodic (B-a.p.) if $u \mapsto u \cdot T$ is a compact operator from $B(G)$ into $\mathrm{VN}(G)$. Let $\mathrm{AP}_B(\widehat{G}) = \{T \in \mathrm{VN}(G) \colon T \text{ is } B\text{-a.p.}\}$. Clearly, $C^*_\delta(G) \subset \mathrm{AP}_B(\widehat{G}) \subset \mathrm{AP}(\widehat{G})$ and it is not hard to show that $\mathrm{AP}_B(\widehat{G}) = \mathrm{AP}(\widehat{G})$ if G is amenable. We will prove in §5 that if G is a closed subgroup of the general Lorentz group $\mathrm{SO}_0(n, 1)$ then $\mathrm{AP}_B(G)$ is contained in $\mathrm{UC}(\widehat{G})$, the norm closure of $A(G) \cdot \mathrm{VN}(G)$ in $\mathrm{VN}(G)$, as defined in Granirer [20]. In particular, $\mathrm{AP}_B(\widehat{G}) = C^*_\delta(G)$ if $G = \mathbf{F}_r$, the free group on r letters. To prove this result, we will first observe that a theorem of de Canniere and Haagerup [4] implies the following: there exists a sequence of continuous positive definite functions p_n on $G = SO_0(n, 1)$ such that $p_n \to 1$ uniformly on compact subsets of G and $p_n \cdot VN(G) \subset UC(\widehat{G})$ for each n. It is interesting to note that if G is a noncompact locally compact group with Kazhdan's property (T) then G does not have such a sequence of positive definite functions.

2. Preliminaries and notations

If E and F are Banach spaces, $\mathscr{B}(E, F)$ will denote the space of bounded linear operators from E to F with the operator norm and $\mathscr{B}(E, E)$ will be written simply as $\mathscr{B}(E)$. The evaluation of a linear functional φ on E at x will be written as $\langle \varphi, x\rangle$ or $\langle x, \varphi\rangle$ and the dual space of E will be denoted by E^*. The $\sigma(E, E^*)$-topology on E will be called the w-topology and the $\sigma(E^*, E)$-topology on E^* the w^*-topology. The inner product in a Hilbert space will be expressed as $(\ ,\ )$.

Thoughout this paper, whenever G is a locally compact group, we will assume that it has a fixed left Haar measure $\mu = \mu_G$; for compact G, we will further assume that $\mu(G) = 1$. Integration with respect to μ will be written as $\int \cdots dx$. If f is a function on G and $x \in G$ then the functions ${}_xf, \check{f}, \tilde{f}$ on G are defined by ${}_xf(y) = f(xy)$, $\check{f}(y) = f(y^{-1})$ and $\tilde{f}(y) = \overline{f(y^{-1})}$, $y \in G$. The left regular representation $\lambda = \lambda_G$ on G is given by $\lambda(x)g(y) = g(x^{-1}y)$, $g \in L^2(G)$, $x, y \in G$. Therefore, if $f \in L^1(G)$, $\lambda(f)(g) = f * g$, $g \in L^2(G)$.

VN(G) is the von Neumann subalgebra of $\mathscr{B}(L^2(G))$ generated by $\{\lambda(x): x \in G\}$. Each $u \in A(G)$ can be written as $h * k^\sim$, $h, k \in L^2(G)$. For $T \in VN(G)$ and $u = h * k^\sim \in A(G)$, $\langle T, \check{u}\rangle = (T(h), k)$, the inner product of $T(h)$ and k in $L^2(G)$. $A(G)$ with pointwise multiplication and the norm $\|u\|_{A(G)} = \inf\{\|h\|_2 \|k\|_2 : u = h * k^\sim,\ h, k \in L^2(G)\}$ is a commutative Banach algebra and $A(G)^* = VN(G)$ (see [17]).

$B(G)$ is the space of coefficient functions of continuous unitary representations of G and let $B_\lambda(G)$ be the space of coefficient functions of continuous unitary representations that are weakly contained in λ_G (see [17]). Then $B(G)$ can be identified with the dual Banach space of $C^*(G)$, the group C^*-algebra of G, and $B_\lambda(G)$ with the dual of $C^*_\lambda(G)$ = the C^*-algebra generated by $\{\lambda(f): f \in L^1(G)\}$. With the respective dual norms and the pointwise multiplications $B(G)$ and $B_\lambda(G)$ are commutative Banach algebras and the $B(G)$-norm equals the $B_\lambda(G)$-norm on $B_\lambda(G)$ (see [17]). Also, $A(G)$ is a closed ideal of $B(G)$ and for $u \in A(G)$, $\|u\|_{A(G)} = \|u\|_{B(G)}$. Let $P(G)$ be the set of continuous positive definite functions on G. Then $B(G)$ is the linear span of $P(G)$ and, for $u \in P(G)$, $\|u\|_{B(G)} = u(e)$, e the identity of G. Let $P_1(G) = \{u \in P(G): \|u\|_{B(G)} = u(e) = 1\}$.

$m \in L^\infty(G)^*$ is called a left invariant mean (LIM) of $L^\infty(G)$ if $\|m\| = m(1_G) = 1$, and $m({}_xf) = m(f)$ if $f \in L^\infty(G)$ and $x \in G$. A locally compact group G is said to be *amenable* if $L^{\infty}(G)$ has a left invariant mean. For example, compact groups and solvable groups are amenable and the (discrete) free group $\mathbf{F}_r$ with r generators, $r > 1$, is not amenable. It is known that, for a locally compact group G, the following three conditions are equivalent: (1) G is amenable; (2) $B_{\lambda}(G) = B(G)$; (3) $A(G)$ has a bounded approximate identity, i.e., a bounded net (u_{α}) in $A(G)$ such that $\lim_{\alpha} \|u_{\alpha}u-u\|_{A(G)} = 0$ for $u \in A(G)$ (see Greenleaf [23] or Pier [40] for these and other properties of amenable groups). A. Lau has shown in [30, Lemma 7.2] that if G is amenable, then there is an approximate identity (p_{α}) of $A(G)$ such that each $p_{\alpha} \in A(G) \cap P_1(G)$. Using this fact it is easy to see that if G is amenable and $T \in \mathrm{VN}(G)$ then $\|T^A\| = \|T\|$. Recall that $T^A \in \mathscr{B}(A(G), \mathrm{VN}(G))$ is defined by $T^A(u) = u \cdot T$ and, in general, $\|T^A\| \le \|T\|$.

Let $\widehat{G}$ be the dual of G, i.e., the equivalence classes of continuous irreducible unitary representations of G with its usual topology (see [10]). For $\sigma \in \widehat{G}$, let d_{σ} be the dimension of σ. G is said to have Kazhdan's property (T) if the trivial representation of G is an isolated point in $\widehat{G}$ (see Kazhdan [29] and Delaroche-Kirillov [9]). For example, compact groups, $\mathrm{SL}(n, \mathbf{R})$ and $\mathrm{SL}(n, \mathbf{Z})$, $n \ge 3$, have property (T); on the other hand, $\mathrm{SL}(2, \mathbf{R})$, $\mathrm{SL}(2, \mathbf{Z})$ and noncompact amenable groups do not have property (T); (see [9]).

A bounded continuous function f on a locally compact group G is almost periodic (weakly almost periodic) if $\{{}_x f : x \in G\}$ is relatively compact (weakly compact). The space of a.p. (w.a.p.) functions on G will be denoted by $\mathrm{AP}(G)$ ($\mathrm{WAP}(G)$). See Burckel [3] for the basic properties of these functions.

If G is abelian and $\widehat{G}$ is its dual than $A(G)$ can be identified with $L^1(\widehat{G})$ (by Fourier transform) and $\mathrm{VN}(G)$ with $L^{\infty}(\widehat{G})$; each $f \in L^{\infty}(\widehat{G})$ can be considered as a multiplication operator on $L^2(\widehat{G})$ which is isomorphic to $L^2(G)$ by Plancherel's theorem. Under these identifications, the module action of $L^1(\widehat{G})$ on $L^{\infty}(\widehat{G})$ is just the usual convolution. It is well known that $\mathrm{UC}(\widehat{G})$, the space of bounded uniformly continuous functions on $\widehat{G}$, equals $L^1(\widehat{G}) * L^{\infty}(\widehat{G})$ and it is proved in [16] that an L^{∞}-function on $\widehat{G}$ is (weakly) almost periodic if and only if the operator $\varphi \mapsto \varphi * f$ from $L^1(\widehat{G})$ into $L^{\infty}(\widehat{G})$ is (weakly) compact. Therefore, for a *general* locally compact group G, Granirer [20] denoted the norm closure of $A(G) \cdot \mathrm{VN}(G)$ by $\mathrm{UC}(\widehat{G})$ and called it the space of uniformly continuous functionals of $A(G)$, and Dunkl-Ramirez [16] called $\{T \in \mathrm{VN}(G) : u \mapsto u \cdot T$ is a (weakly) compact operator from $A(G)$ into $\mathrm{VN}(G)\}$ the space of (weakly) almost periodic functionals on $A(G)$ and denoted it by ($\mathrm{WAP}(\widehat{G})$) $\mathrm{AP}(\widehat{G})$. In this paper, we will use their terminologies.

Lemma 2.1. *Let G be a locally compact group.*

(1) *$\mathrm{AP}(\widehat{G})$ and $\mathrm{WAP}(\widehat{G})$ are selfadjoint closed $B(G)$-submodules of $\mathrm{VN}(G)$.*
(2) *If $x \in G$, then $\lambda(x)\mathrm{AP}(\widehat{G}) \subset \mathrm{AP}(\widehat{G})$ and $\lambda(x)\mathrm{WAP}(\widehat{G}) \subset \mathrm{WAP}(\widehat{G})$.*

See [16] and [20] for (1). (2) was proved in [6]. It is also known that $\mathrm{UC}(\widehat{G})$ is a C^*-algebra (see [21]). But it is not known in general whether $\mathrm{AP}(\widehat{G})$ and $\mathrm{WAP}(\widehat{G})$ are C^*-algebras.

For a locally compact group G, we will denote the linear span of $\{\lambda(x): x \in G\}$ by $\mathrm{Trig}\,\widehat{G}$. (Thus, $C^*_\delta(G)$ is the norm closure of $\mathrm{Trig}\,\widehat{G}$ in $\mathrm{VN}(G)$.) This is motivated by the fact that if G is abelian then $\sum_{k=1}^n c_k\lambda(x_k)$ can be identified with the trigonometric polynomial $\gamma \mapsto \sum_{k=1}^n c_k\overline{\gamma(x_k)}$ on $\widehat{G}$. It is well known that if G is abelian then $\mathrm{Trig}\,\widehat{G}$ is uniformly dense in $\mathrm{AP}(\widehat{G})$ (as function spaces on $\widehat{G}$), i.e., $\mathrm{AP}(\widehat{G}) = C^*_\delta(G)$. For $G = \mathbf{R}$ and $\mathbf{T}$, this uniform approximation theorem was first proved by H. Bohr, the creator of the theory of almost periodic functions (see [2]). Therefore, we would like to give the following.

Definition. A locally compact group G is said to have the *dual Bohr approximation property* if $\mathrm{AP}(\widehat{G}) = C^*_\delta(G)$.

If $u \in A(G)$ and $x \in G$ then $u \cdot \lambda(x) = u(x)\lambda(x)$ (see [17]). Therefore, the operator $\lambda(x)^A$ is of rank one. Thus if $T \in \mathrm{Trig}\,\widehat{G}$ then T^A is of finite rank and hence $C^*_\delta(G) \subset \mathrm{AP}(\widehat{G})$ (see [16] and [30]). Conversely, if $T \in \mathrm{VN}(G)$ and T^A is of finite rank then $T \in \mathrm{Trig}\,\widehat{G}$. To prove this fact and for later citations, we will first summarize some of the known results of Eymard [17] concerning the supports of operators in $\mathrm{VN}(G)$ as a lemma. Recall that for $T \in \mathrm{VN}(G)$, $\mathrm{supp}\,T$ (the support of T) is the closed set of all x in G such that whenever $u \in A(G)$ and $u(x) \neq 0$ then $u \cdot T \neq 0$.

Lemma 2.2 (Eymard [17]). *Let G be a locally compact group, $T \in \mathrm{VN}(G)$.*

(1) *If* $\mathrm{supp}\,T = \varnothing$, *then* $T = 0$; $\mathrm{supp}\,T = \{x\}$ *if and only if* $T = c\lambda(x)$ *for some nonzero* $c \in \mathbf{C}$.
(2) *If* $v \in B(G)$ *then* $\mathrm{supp}(v \cdot T) \subset \mathrm{supp}\,v \cap \mathrm{supp}\,T$; *if* $T_1, T_2 \in \mathrm{VN}(G)$ *then* $\mathrm{supp}(T_1T_2) \subset (\mathrm{supp}\,T_1)(\mathrm{supp}\,T_2)$.
(3) *If* $v \in B(G)$ *and* v *is identically equal to* 1 *on a neighborhood of* $\mathrm{supp}\,T$ *then* $v \cdot T = T$.
(4) *If* K *is a closed subset of* G *and* (T_α) *a net in* $\mathrm{VN}(G)$ *such that* $\mathrm{supp}\,T_\alpha \subset K$ *and* $w^*\text{-}\lim_\alpha T_\alpha = T \in \mathrm{VN}(G)$ *then* $\mathrm{supp}\,T \subset K$.

(1), (2) and (4) are contained in [17, pp. 225–229], (3) is a direct consequence of (1) and (2).

Proposition 2.3. *Let G be a locally compact group and $T \in \mathrm{VN}(G)$. If T^A is a finite rank operator then $T \in \mathrm{Trig}\,\widehat{G}$.*

Proof. Note first that if $T \in \mathrm{VN}(G)$ and $y_1, \ldots, y_n$ are n distinct elements in supp T then the dimension of the linear space $\{u \cdot T: u \in A(G)\}$ is at least n. Indeed, by Eymard [17, Lemma 3.2] there exist $u_1, \ldots, u_n$ in $A(G)$ such that

$$(*) \qquad u_i(x_i) = 1 \quad \text{for each } i \qquad \text{and} \qquad \mathrm{supp}\,u_i \cap \mathrm{supp}\,u_j = \varnothing \quad \text{if } i \neq j.$$

Suppose that $\sum_{j=1}^n c_j(u_j \cdot T) = 0$ where $c_j \in \mathbf{C}$. Then

$$0 = u_i \cdot \left(\sum_{j=1}^n c_j(u_j \cdot T) \right) = c_i u_i^2 \cdot T, \qquad i = 1, \dots, n.$$

Since $u_i(x_i) = 1$ and $x_i \in \operatorname{supp} T$, $u_i^2 \cdot T \neq 0$. Therefore $c_i = 0$ for all i and hence $\{u_i \cdot T : i = 1, \dots, n\}$ is linearly independent.

Now suppose that T^A is of finite rank. Then by the above paragraph, supp T is finite, say, $\operatorname{supp} T = \{x_1, \dots, x_n\}$. Choose $u_1, \dots, u_n$ in $A(G)$ such that (*) holds and, in addition, $u_i = 1$ on a neighborhood of x_i. Then $u_1 + \cdots + u_n = 1$ on a neighborhood of supp T. By 2.2(3), $T = (u_1 + \cdots + u_n) \cdot T = \sum_{i=1}^n u_i \cdot T$. But by 2.2(2),

$$\operatorname{supp}(u_i \cdot T) \subset \operatorname{supp} u_i \cap \operatorname{supp} T = \{x_i\}.$$

Thus, by 2.2(1), $u_i \cdot T = c_i \lambda(x_i)$ and hence $T = \sum_{i=1}^n c_i \lambda(x_i) \in \operatorname{Trig} \widehat{G}$.

Remark. Let $\mathrm{VN}^A(G) = \{T^A : T \in \mathrm{VN}(G)\} \subset \mathscr{B}(A(G), \mathrm{VN}(G))$. Note that for $T \in \mathrm{VN}(G)$, $\|T^A\| \leq \|T\|$. Therefore, if G has the dual Bohr approximation property that each compact operator in $\mathrm{VN}^A(G)$ can be approximated in the $\mathscr{B}(A(G), \mathrm{VN}(G))$-norm by the finite rank operators in $\mathrm{VN}^A(G)$. If G is amenable, for $T \in \mathrm{VN}(G)$, $\|T^A\| = \|T\|$. Therefore, by the above proposition, to ask whether an amenable locally compact group G has the dual Bohr approximation property is the same as to ask whether the compact operators in $\mathrm{VN}^A(G)$ can be approximated in the $\mathscr{B}(A(G), \mathrm{VN}(G))$-norm by the finite rank operators in $\mathrm{VN}^A(G)$.

To conclude this section, we would like to quote the following known result for later citations.

Theorem 2.4. *Let (p_α) be the net of continuous positive definite functions on a locally compact group G such that $\lim_\alpha p_\alpha = p$ uniformly on compacta. Then $\lim_\alpha \|p_\alpha u - pu\|_{A(G)} = 0$ for each $u \in A(G)$.*

The above theorem is a special case of Theorem 5.5 of McKennon [34] which in turn is a special case of Theorem B_2 of Granirer and Leinert [45]. In our opinion, the proof of Theorem B_2 in [45] is more transparent than the corresponding proof in [34]. An independent proof of Theorem 2.4, attributed to P. Nielson, is contained in the appendix of deCanniere and Haagerup [4].

3. The rank one operator E_G for compact G

In this section we will only consider compact groups. We will first outline some of the basic results in harmonic analysis on compact groups (for details see either Dunkl and Ramirez [12] or Hewitt and Ross [25]). Let $\widehat{G}$ be the dual of a compact group G. For each $\sigma \in \widehat{G}$, choose $\pi_\sigma \in \sigma$. Then d_σ, the dimension of π_σ, is finite and we will denote the character of π_σ by χ_σ: $\chi_\sigma(x) = \operatorname{Tr}(\pi_\sigma(x))$, $x \in G$. $\pi_\sigma(x)$ will be considered as a $d_\sigma \times d_\sigma$ unitary matrix and the (ij)th component of $\pi_\sigma(x)$ will be denoted by u^σ_{ij}. The linear span of u^σ_{ij}, $1 \leq i, j \leq d_\sigma$, is a d^2_σ-dimensional subspace V_σ of $A(G) \cap L^2(G)$ and it is closed under convolution on the left and on the right by L^1-functions. By the Peter-Weyl theorem, $\sum\{V_\sigma : \sigma \in \widehat{G}\} = L^2(G)$ (Hilbert sum). If $f \in L^1(G)$, then its Fourier transform is given by $\hat{f}(\sigma) = \int_G f(x)\pi_\sigma(x^{-1})\,dx$, $\sigma \in \widehat{G}$. The operator norm of $\lambda(f)$ is $\|\lambda(f)\| = \sup\{\|\hat{f}(\sigma)\| : \sigma \in \widehat{G}\}$ where $\|\hat{f}(\sigma)\|$ is the operator norm of $\hat{f}(\sigma)$ on $\mathbf{C}^{d_\sigma}$. If $u \in A(G)$ then $\sum_{\sigma \in \widehat{G}} d_\sigma(\chi_\sigma * u)$ converges uniformly to u and $\|u\|_{A(G)} = \sum_\sigma d_\sigma \|\hat{u}(\sigma)\|_1$, where $\|\hat{u}(\sigma)\|_1 = \mathrm{Tr}(|\hat{u}(\sigma)|)$. In particular, $\|u\|_{A(G)} \geq d_\sigma \|\hat{u}(\sigma)\|_1 \geq d_\sigma \|\hat{u}(\sigma)\|$ for each $\sigma \in \widehat{G}$, and hence

$$\text{(3.1)} \qquad \frac{1}{d_\sigma}\|u\|_{A(G)} \geq \|\hat{u}(\sigma)\|, \qquad \sigma \in \widehat{G}.$$

Definition (McMullen and Price [35]). A compact group G is said to be *tall* if, for each positive integer n, the set $\{\sigma \in \widehat{G} : d_\sigma = n\}$ is finite.

If G is an infinite abelian group then for each $\sigma \in \widehat{G}$, $d_\sigma = 1$ and hence G is not tall. On the other hand, it is well known that compact semisimple Lie groups are tall (see Hutchinson [27]).

If G is an infinite compact abelian group then $\mathrm{AP}(\widehat{G}) \cap C^*_\lambda(G) = (0)$, since $C^*_\lambda(G) \approx C_0(\widehat{G})$, the functions on $\widehat{G}$ that are vanishing at infinity (cf. [12]); in particular, $E = \lambda(1_G) \notin \mathrm{AP}(\widehat{G}) = C^*_\delta(G)$. Dunkl and Ramirez proved in [15, Theorem 11] that if G is a compact tall group then $E \in \mathrm{AP}(\widehat{G})$. This is the "if" part of the following proposition, for the sake of completeness, we will also include its proof here.

Proposition 3.1. *Let G be a compact group. Then $E \in \mathrm{AP}(\widehat{G})$ if and only if G is a tall group.*

Proof. (1) (Dunkl and Ramirez [15]) Assume that G is tall. Let

$$S_n \in \mathscr{B}(A(G), \mathrm{VN}(G))$$

be given by

$$S_n(u) = \lambda\left(\sum\{d_\sigma(\chi_\sigma * u) : \sigma \in \widehat{G}, d_\sigma \leq n\}\right).$$

Since $\chi_\sigma * u \in V_\sigma$ and G is tall, S_n is a finite rank operator. For $u \in A(G)$,

$$\begin{aligned}\|(E^A - S_n)(u)\| &= \|u \cdot \lambda(1_G) - S_n(u)\| = \left\|\lambda(u) - \lambda\left(\sum_{d_\sigma \leq n} d_\sigma(\chi_\sigma * u)\right)\right\| \\ &= \left\|\lambda\left(\sum_{d_\sigma > n} d_\sigma(\chi_\sigma * u)\right)\right\| = \sup\{\|\hat{u}(\sigma)\| : d_\sigma > n\} \\ &< \frac{1}{n}\|u\|_{A(G)}, \quad \text{by (3.1)}.\end{aligned}$$

Thus, $\lim_n \|E^A - S_n\| = 0$, and hence E^A is a compact operator, i.e., $E \in \mathrm{AP}(\widehat{G})$.

(2) Assume that G is not tall. Then there exists a positive integer n_0 such that $Q = \{\sigma \in \widehat{G} : d_\sigma = n_0\}$ is infinite. Note that if $\sigma \in \widehat{G}$ then $\hat{\chi}_\sigma(\sigma) = (1/d_\sigma) I_{d_\sigma}$ and $\hat{\chi}_\sigma(\beta) = 0$ if $\beta \neq \sigma$ where I_{d_σ} denotes the $d_\sigma \times d_\sigma$ identity matrix (cf. [25, p. 80]). Therefore, if $\sigma \in Q$,

$$\|\chi_\sigma\|_{A(G)} = d_\sigma \cdot \frac{1}{d_\sigma} \|I_{d_\sigma}\|_1 = d_\sigma = n_0 .$$

On the other hand, if $\sigma, \sigma' \in Q$, $\sigma \neq \sigma'$, then

$$\begin{aligned} \|E^A(\chi_\sigma) - E^A(\chi_{\sigma'})\| &= \|\lambda(\chi_\sigma) - \lambda(\chi_{\sigma'})\| \\ &= \max\left\{ \frac{1}{d_\sigma}\|I_{d_\sigma}\|, \frac{1}{d_{\sigma'}}\|I_{d_{\sigma'}}\| \right\} = \frac{1}{n_0} . \end{aligned}$$

So the image under E^A of the bounded subset Q of $A(G)$ is not relatively compact, i.e., $E \notin \mathrm{AP}(\widehat{G})$.

Remark. When G is an infinite compact tall group, the above proposition shows that the operator E^A can be approximated in norm by the finite rank operators S_n in $\mathscr{B}(A(G), \mathrm{VN}(G))$. But none of the operators S_n belongs to $\mathrm{VN}^A(G)$. Indeed, if there exists n such that $S_n = T^A$ for some $T \in \mathrm{VN}(G)$ then T^A is of finite rank. By Proposition 2.3., $T = \sum_{i=1}^n c_i \lambda(x_i) \in \operatorname{Trig} \widehat{G}$. Since $S_n \neq 0$, there exists $u \in A(G)$ such that $S_n(u) = T^A(u) = u \cdot T \neq 0$. But $S_n(u) \in \{\lambda(f) : f \in L^1(G)\}$ and $u \cdot T = \sum_{i=1}^n c_i u(x_i) \lambda(x_i) \in \operatorname{Trig} \widehat{G}$. This is impossible, since $\operatorname{Trig} \widehat{G} \cap \lambda(L^1(G)) = (0)$ for any nondiscrete locally compact group G.

Definition (Rosenblatt [42]). A compact group G is said to have the *mean-zero weak containment* property if there exists a net $\{g_\alpha\}$ in $L_0^2(G) = \{f \in L^2(G) : \int f(x)\,dx = 0\}$ such that $\|g_\alpha\|_2 = 1$ and $\lim_\alpha \|\lambda(x) g_\alpha - g_\alpha\|_2 = 0$ for each $x \in G$; in other words, if the left regular representation of G_d (the discrete version of G) on $L_0^2(G)$ weakly contains the trivial representation of G_d.

It is known that if G_d is amenable then G has this property (cf. Rosenblatt [41]). On the other hand, if G contains a dense subgroup H such that H_d has Kazhdan's property (T), e.g., $G = \mathrm{SO}(n)$, $n \geq 5$, then G does not have the mean-zero weak containment property (see Margulis [35] and Sullivan [44]).

Proposition 3.2 (Chou, Lau and Rosenblatt [7]). *An infinite compact group G has the mean-zero weak containment property if and only if $E \notin C_\delta^*(G)$.*

Therefore, as was concluded in [7], $E \in C_\delta^*(G)$ if $G = \mathrm{SO}(n)$, $n \geq 5$.

Lemma 3.3. *A compact group G has the mean-zero weak containment property if and only if there exists a net $\{g_\alpha\}$ in $L_0^2(G)$ such that $\|g_\alpha\|_2 = 1$ and $\lambda(x) g_\alpha - g_\alpha \to 0$ weakly in $L_0^2(G)$ for each $x \in G$.*

Proof. Applying the well-known argument of the proof of Theorem 2.2 of Namioka [37], the result follows easily.

Proposition 3.4. *Let* $G = \prod_{n=1}^{\infty} G_n$ *be the (complete) direct product of nontrivial compact groups* G_n. *Then* G *has the mean-zero weak containment property.*

Proof. Each $x \in G$ can be written as $x = (x^n)$, $x^n \in G_n$. Thus a function on G can be considered as a function in infinitely many variables x^n, $x^n \in G_n$. By the above lemma, to show that G has the mean-zero weak containment property it suffices to show that given $x_1, \dots, x_l$ in G, $h_1, \dots, h_m$ in $L_0^2(G)$ with $\|h_j\|_2 = 1$ for all j, and $\varepsilon > 0$, then there exists $f \in L_0^2(G)$, $\|f\|_2 = 1$ such that

$$|(\lambda(x_i)f - f, h_j)| < \varepsilon \quad \text{for } i = 1, \dots, l, \ j = 1, \dots, m.$$

The Haar measure μ of G is the product of the Haar measures μ_{G_n}, $n = 1, 2, \dots$, and the collection of functions in $L_0^2(G)$ that depend only on finitely many of the variables x^n is dense in $L_0^2(G)$. Therefore, there are functions $k_1, \dots, k_m$ with $k_j \in L_0^2(G)$, $\|k_j\|_2 = 1$ and a positive integer N such that the functions k_j only depend on the first N variables $x^1, \dots, x^N$ and

$$(*) \qquad \|h_j - k_j\|_2 < \varepsilon/2, \qquad j = 1, \dots, m.$$

For $i = 1, \dots, l$, let $\overline{x}_i = (x_i^1, \dots, x_i^N, e, e, \dots)$. Then $\lambda(x_i^{-1})k_j = \lambda(\overline{x}_i^{-1})k_j$. Thus, for $f \in L_0^2(G)$,

$$(**) \qquad \begin{aligned} (\lambda(x_i)f - f, k_j) &= (f, \lambda(x_i^{-1})k_j - k_j) \\ &= (f, \lambda(\overline{x}_i^{-1})k_j - k_j) = (\lambda(\overline{x}_i)f - f, k_j). \end{aligned}$$

Take $g \in L_0^2(G_{N+1})$ such that $\|g\|_{L^2(G_{N+1})} = 1$. (Such g exists, since $G_{N+1} \neq \{e\}$.) For $x \in G$, let $f(x) = g(x^{N+1})$. Then $f \in L_0^2(G)$ and $\|f\|_2 = 1$ and $\lambda(\overline{x}_i)f = f$ for $i = 1, \dots, l$, and hence, by ($**$), $(\lambda(x_i)f - f, k_j) = 0$. Thus, by ($*$)

$$|(\lambda(x_i)f - f, h_j)| = |(\lambda(x_i)f - f, h_j - k_j)| < \varepsilon,$$

$i = 1, \dots, l$, $j = 1, \dots, m$. The proof is complete.

According to Hutchinson [27, 28], the following products of compact groups are tall:

(1) $G_1 = \prod_{n=2}^{\infty} \mathrm{SU}(n)$,
(2) $G_2 = \prod_{n=3}^{\infty} \mathrm{SO}(n)$,
(3) $G_3 = \prod_{n=5}^{\infty} A_n$ where A_n is the alternating group on n letters.

Therefore, by Propositions 3.1, 3.2 and 3.4, we have the following.

Theorem 3.5. *There exist infinite tall compact groups that have the mean-zero weak containment property. If* G *is such a group then* $E \in \mathrm{AP}(\widehat{G}) \setminus C_\delta^*(G)$ *and hence* G *does not have the dual Bohr approximation property.*

Remarks. (1) Let $G = G_1$, G_2 or G_3. Then, by the proof of Proposition 3.1, the operator $E^A \in \mathscr{B}(A(G), \mathrm{VN}(G))$ can be approximated in norm by the finite rank operators in $\mathscr{B}(A(G), \mathrm{VN}(G))$ but it cannot be approximated in norm by the finite rank operators in $\mathrm{VN}^A(G)$.

(2) The groups G_i, $i = 1, 2, 3$, are not amenable as discrete groups. G_1 and G_2 are not amenable as discrete groups, since SO(3) and SU(2) already contain $\mathbf{F}_2$ as a subgroup (cf. Greenleaf [23]). To see that G_3 is not amenable as a discrete group we will first prove that $S = \prod_{n=5}^{\infty} S_n$ is not amenable as discrete, where S_n is the symmetric group on n letters. Indeed, since $\mathbf{F}_2$ is residually finite (cf. [32]), there exist finite groups K_j such that $\mathbf{F}_2$ can be embedded into $\prod_{j=1}^{\infty} K_j$. Now each K_j can be considered as a subgroup of S_{n_j} for some $n_j \geq 5$ and $n_1 < n_2 < \cdots$. Therefore $\prod_{j=1}^{\infty} K_j$ can be considered as a subgroup of S and hence S contains a copy of $\mathbf{F}_2$. Thus S is not amenable as discrete. Note that G_3 is a normal subgroup of S and $S/G_3 \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \cdots$ is abelian and hence is amenable as discrete. Thus G_3 is not amenable as a discrete; otherwise, S being the extension of an amenable as discrete group G_3 by an abelian group would be amenable as discrete (cf. [23]). We do not yet know whether there exists an infinite tall compact group G such that G_d is amenable.

(3) G_1 and G_2 are connected groups, and G_3 is a totally disconnected group, or equivalently, a profinite group. These three groups are, of course, not Lie groups. In fact, we will prove later on in this section that if G is a compact Lie group then $E \in C_\delta^*(G)$ if and only if $E \in \mathrm{AP}(\widehat{G})$.

Let G_1 and G_2 be locally compact groups. Then the Hilbert space tensor product of $L^2(G_1)$ and $L^2(G_2)$ can be identified with $L^2(G_1 \times G_2)$: if $h_i \in L^2(G_i)$, $i = 1, 2$, then $h_1 \otimes h_2 \in L^2(G_1 \times G_2)$ is given by $(h_1 \otimes h_2)(x, y) = h_1(x)h_2(y)$, $(x, y) \in G_1 \times G_2$. If $S \in \mathscr{B}(L^2(G_1))$ and $T \in \mathscr{B}(L^2(G_2))$, $S \otimes T \in \mathscr{B}(L^2(G_1 \times G_2))$ is defined by $(S \otimes T)(h_1 \otimes h_2) = S(h_1) \otimes S(h_2)$, $h_i \in L^2(G_i)$, $i = 1, 2$. It is well known that $\|S \otimes T\| \leq \|S\|\,\|T\|$. The (spatial) C^*-algebra tensor product of $C_\delta^*(G_1)$ and $C_\delta^*(G_2)$, denoted by $C_\delta^*(G_1) \otimes C_\delta^*(G_2)$, is the operator norm closure of the linear span of $\{S \otimes T\colon S \in C_\delta^*(G_1),\ T \in C_\delta^*(G_2)\}$.

Lemma 3.6. *Let G_1 and G_2 be locally compact groups. Then $C_\delta^*(G_1) \otimes C_\delta^*(G_2) = C_\delta^*(G_1 \times G_2)$. In particular, if G_1 and G_2 are compact and $E_{G_i} \in C_\delta^*(G_i)$, $i = 1, 2$, then $E_{G_1 \times G_2} \in C_\delta^*(G_1 \times G_2)$.*

Proof. Let $S \in C_\delta^*(G_1)$ and $T \in C_\delta^*(G_2)$. Then there exist $S_n \in \operatorname{Trig} \widehat{G}_1$ and $T_n \in \operatorname{Trig} \widehat{G}_2$ such that $S_n \to S$ and $T_n \to T$ in norm. Consequently,

$$\begin{aligned} \|S_n \otimes T_n - S \otimes T\| &\leq \|S_n \otimes (T_n - T)\| + \|(S_n - S) \otimes T\| \\ &\leq \|S_n\|\,\|T_n - T\| + \|S_n - S\|\,\|T\| \to 0, \end{aligned}$$

as $n \to 0$. Since, for each n, $S_n \otimes T_n \in \operatorname{Trig}(G_1 \times G_2)^\wedge$, we conclude that $S \otimes T \in C_\delta^*(G_1 \times G_2)$ and hence $C_\delta^*(G_1) \otimes C_\delta^*(G_2) \subset C_\delta^*(G_1 \times G_2)$. Conversely, for any $(x, y) \in G_1 \times G_2$, $\lambda_{G_1 \times G_2}(x, y) = \lambda_{G_1}(x) \otimes \lambda_{G_2}(y) \in C^*_\delta(G_1) \otimes C^*_\delta(G_2)$. Therefore, $C^*_\delta(G_1 \times G_2) \subset C^*_\delta(G_1) \otimes C^*_\delta(G_2)$.

Now assume that G_1 and G_2 are compact and $E_{G_i} \in C^*_\delta(G_i)$, $i = 1, 2$. Then $E_{G_1 \times G_2} = E_{G_1} \otimes E_{G_2} \in C^*_\delta(G_1) \otimes C^*_\delta(G_2)$ which equals $C^*_\delta(G_1 \times G_2)$ by the above paragraph. Thus the second statement of the lemma holds.

Lemma 3.7. *Let G be a compact group and F a finite normal subgroup of G. Then $E_G \in C^*_\delta(G)$ if and only if $E_{G/F} \in C^*_\delta(G/F)$.*

Proof. That $E_G \in C^*_\delta(G)$ implies $E_{G/F} \in C^*_\delta(G/F)$ follows from Proposition 2.8 of [7].

To prove the converse, assume that $E_{G/F} \in C^*_\delta(G/F)$. Let $F = \{t_1, \ldots, t_N\}$ and for $x \in G$ the coset xF will be written as $\dot{x}$. Then for $f \in L^1(G)$,

$$\int_G f(x)\,dx = \int_{G/F} \frac{1}{N} \sum_{j=1}^{N} f(t_j^{-1}x)\,d\dot{x}.$$

Let $\varepsilon > 0$ be given. Then, by assumption, there exist $x_1, \ldots, x_n \in G$ and $c_1, \ldots, c_n \in \mathbf{C}$ such that

$$(*) \qquad \left\| E_{G/F} - \sum_{i=1}^{n} c_i \lambda_{G/F}(\dot{x}_i) \right\| \le \varepsilon.$$

Let

$$T = \frac{1}{N} \sum_{i=1}^{n} \sum_{j=1}^{N} c_i \lambda_G(x_i t_j) \in \operatorname{Trig} \widehat{G}.$$

To finish the proof of this lemma, it suffices to show that $\|E_G - T\| \le \varepsilon$. Let $f \in L^2(G)$ and let $g(\dot{x}) = \frac{1}{N} \sum_{j=1}^{N} f(t_j^{-1}x)$, $\dot{x} \in G/F$. Then $g \in L^2(G/F)$, $\|g\|_{L^2(G/F)} \le \|f\|_{L^2(G)}$ and we have

$$\begin{aligned}
&\|E_G(f) - T(f)\|^2_{L^2(G)} \\
&= \int_G \left| \left(\int_{G/F} g(\dot{x})\,d\dot{x} \right) \cdot 1_G(y) - \sum_{i=1}^{m} c_i \frac{1}{N} \sum_{j=1}^{N} f(t_j^{-1} x_i^{-1} y) \right|^2 dy \\
&= \int_{G/F} \left| E_{G/F}(g)(\dot{y}) - \sum_{i=1}^{n} c_i g(\dot{x}_i^{-1} \dot{y}) \right|^2 d\dot{y} \\
&= \left\| E_{G/F}(g) - \sum_{i=1}^{n} c_i \lambda_{G/F}(\dot{x}_i)(g) \right\|^2_{L^2(G/F)} \\
&\le \left\| E_{G/F} - \sum_{i=1}^{n} c_i \lambda_{G/F}(\dot{x}_i) \right\|^2 \|g\|^2_{L^2(G/F)} \\
&\le \varepsilon^2 \|g\|^2_{L^2(G/F)} \quad (\text{by } (*)) \\
&\le \varepsilon^2 \|f\|^2_{L^2(G)}.
\end{aligned}$$

Thus $\|E_G - T\| \le \varepsilon$, as claimed.

According to Margulis [33, Proposition 4] (see also Sullivan [44]), if a connected compact simple Lie group G is not locally isomorphic to SO(3) or SO(4) then G contains a dense subgroup H such that H_d has Kazhdan's property (T), and hence as noted by Margulis [33] and Sullivan [44], independently, G does not have the mean-zero weak containment property. More recently, Drinfel′d [11] has shown that SO(3) and SO(4) do not have the mean-zero weak containment property. Combining these results together with Proposition 3.2 and Lemma 3.7, we can conclude the following.

Proposition 3.8. *Let G be a connected compact simple Lie group. Then $E_G \in C^*_\delta(G)$.*

Remark. It is interesting to compare Drinfel′d's result with an unpublished result of Margulis that SO(3) and SO(4) do not contain a dense subgroup with property (T). Note that it is well known that SO(3) is locally isomorphic to SU(2) and SO(4) is locally isomorphic to $SO(3) \times SO(3)$.

Granirer [19] and Rudin [43] have proved, independently, that if G is an infinite compact group and G_d is amenable then $L^\infty(G)$ has more than one left invariant mean. On the other hand, it is known that if G is compact and $E_G \in C^*_\delta(G)$ then $L^\infty(G)$ has a unique left invariant mean (see [7]).

Lemma 3.9. *Suppose that F is a finite normal subgroup of a compact group G. If $L^\infty(G/F)$ has a unique left invariant mean then so does $L^\infty(G)$.*

Proof. Let θ be the natural homomorphism from G onto G/F and let $F = \{t_1, \dots, t_N\}$. If m is a LIM on $L^\infty(G)$, define $\overline{m} \in L^\infty(G/F)^*$ by $\overline{m}(g) = m(g \circ \theta)$, $g \in L^\infty(G/F)$. It is easily checked that $\overline{m}$ is a LIM on $L^\infty(G)$.

Let m_1 and m_2 be LIM's on $L^\infty(G)$. Since $L^\infty(G/F)$ has a unique LIM, $\overline{m}_1 = \overline{m}_2$. for $f \in L^\infty(G)$, let $g(\dot{x}) = \frac{1}{N}\sum_{j=1}^N f(t_j x)$, $\dot{x} \in G/F$. Then $g \in L^\infty(G/F)$ and

$$(g \circ \theta)(x) = \frac{1}{N}\sum_{j=1}^N f(t_j x), \qquad x \in G,$$

i.e., $g \circ \theta = \frac{1}{N}({}_{t_1}f + \cdots + {}_{t_N}f)$. Thus, for $f \in L^\infty(G)$,

$$\begin{aligned} m_1(f) &= \frac{1}{N} m_1({}_{t_1}f + \cdots + {}_{t_N}f) = m_1(g \circ \theta) = \overline{m}_1(g) \\ &= \overline{m}_2(g) = m_2(g \circ \theta) = \frac{1}{N} m_2({}_{t_1}f + \cdots + {}_{t_N}f) = m_2(f). \end{aligned}$$

Therefore, $m_1 = m_2$, i.e. $L^\infty(G)$ has a unique LIM.

Theorem 3.10. *Let G be a compact Lie group. Then the following conditions are equivalent:*

(1) $E_G \in C^*_\delta(G)$.

(2) *G does not have the mean-zero containment property.*

(3) $L^\infty(G)$ *has a unique left invariant mean.*
(4) G *is semisimple.*
(5) G *is tall.*
(6) $E_G \in \mathrm{AP}(\widehat{G})$.

Proof. $(1) \Leftrightarrow (2)$, $(5) \Leftrightarrow (6)$ and $(2) \Rightarrow (3)$ are true for general compact groups (see Proposition 3.2, Proposition 3.1 and [7, Proposition 1.3], respectively). $(4) \Leftrightarrow (5)$ is contained in Hutchinson [27]. It remains to show $(4) \Rightarrow (1)$ and $(3) \Rightarrow (4)$. Since G is a compact Lie group, the connected component of G is an open normal subgroup of G of finite index. Therefore, to prove both implications we may assume that G is connected.

$(4) \Rightarrow (1)$ Let G be a connected compact semisimple Lie group and G^* the universal covering group of G. Then there exists a finite central group F of G^* such that G^*/F is isomorphic to G (see Hochschild [26, p. 144]]). Since G^* is simply connected, it is well known that G^* is a direct product of connected compact simple Lie groups $G_1, \ldots, G_n$. By Proposition 3.8, $E_{G_i} \in C^*_\delta(G_i)$ for $i = 1, \ldots, n$. Therefore, by Lemma 3.6, $E_{G^*} \in C^*_\delta(G^*)$ and hence by Lemma 3.7, $E_G \in C^*_\delta(G)$.

$(3) \Rightarrow (4)$ Let G be a connected nonsemisimple compact Lie group. Then there exist a connected compact Lie group G^* and a finite normal subgroup F of G^* such that $G^*/F \cong G$ and $G^* = \mathbf{T}^k \otimes H$ where $k \geq 1$ and H is semisimple (see [26, p. 144]). Since $L^\infty(\mathbf{T}^k)$ has more than one LIM, so does $L^\infty(G^*)$ (see [5, p. 49]). By Lemma 3.9, $L^\infty(G)$ also has more than one LIM.

Remark. We have shown earlier that there exist both connected compact groups and totally disconnected compact groups that satisfy (6) but not (1). We do not know whether the following implications are true for a general compact group: $(3) \Rightarrow (5)$, $(5) \Rightarrow (3)$, $(3) \Rightarrow (2)$, but by applying a result in Hutchinson [28] we are able to show that if G is a totally disconnected compact group then $(3) \Rightarrow (5)$.

Proposition 3.11. *Let* G *be a nontall totally disconnected compact group. Then* $L^\infty(G)$ *has more than one left invariant mean.*

Proof. By Theorem 3.2 of Hutchinson [28], G has an open normal subgroup H and a closed normal subgroup K such that $K \subset H$ and either (1) H/K is an infinite abelian group or (2) H/K is a product of infinitely many copies of a fixed finite simple group S. In either case, H/K is an infinite compact group and $(H/K)_d$ is amenable (see [13, Theorem 3.8] for the fact that the product of infinitely many copies of the same finite group is locally finite and hence is amenable as discrete). Therefore, by Granirer [19] or Rudin [43], $L^\infty(H/K)$ has more than one LIM and hence, by [5, p. 49], $L^\infty(H)$ has more than one LIM. Finally it is well known that the set of LIM's on $L^\infty(H)$ can be embedded into the set of LIM's on $L^\infty(G)$. Thus $L^\infty(G)$ also has more than one LIM.

4. Extensions of locally compact groups by discrete groups

The only known groups with the dual Bohr approximation property are locally compact abelian groups and amenable discrete groups. In this section we will show that any extension of a locally compact abelian group by an amenable discrete group also has this property.

Let H be an open subgroup of a locally compact group G. The left Haar measure μ_H of H is taken to be the restriction of μ_G to H. If f is a function on H then $\mathring{f}$ will denote the function on G given by $\mathring{f}(x) = f(x)$ if $x \in H$, and $\mathring{f}(x) = 0$ if $x \in G \backslash H$. It is known that $u \to \mathring{u}$ is an isometry of $A(H)$ into $A(G)$ (cf. Eymard [17, p. 215]). As in [17], we will denote the von Neumann subalgebra generated by $\{\lambda_G(x) \colon x \in H\}$ by $\mathrm{VN}_H(G)$. Eymard [17, p. 215] has shown that the mapping $v \to v|H$ (the restriction of v to H) is a mapping of $A(G)$ onto $A(H)$ and its transpose, $T \to \mathring{T}$, is a von Neumann algebra isomorphism of $\mathrm{VN}(H)$ onto $\mathrm{VN}_H(G)$. The following simple lemma is essentially known; we include it here for the sake of completeness.

Lemma 4.1. *Let H be an open subgroup of a locally compact group G. Then* $\mathrm{VN}_H(G) = \{T \in \mathrm{VN}(G) \colon \operatorname{supp} T \subset H\}$.

Proof. If $T \in \mathrm{VN}_H(G)$ then there is a net (T_α) such that each T_α is a linear combination of members of $\{\lambda_G(x) \colon x \in H\}$ and $T_\alpha \to T$ in the w^*-topology. Thus $\operatorname{supp} T_\alpha$ is a finite subset of the closed set H and hence by Lemma 2.2(4), $\operatorname{supp} T \subset H$.

Conversely, if $T \in \mathrm{VN}(G)$ and $\operatorname{supp} T \subset H$ then, by Lemma 2.2(3) $\chi_H \cdot T = T$ where χ_H is the characteristic function of H in G. Choose a net (T_α) in $\operatorname{Trig} \widehat{G}$ such that $T_\alpha \to T$ in the w^*-topology. Then $\chi_H \cdot T_\alpha$ converges to $\chi_H \cdot T$ in the w^*-topology. Since each T_α is of the form $\sum_{i=1}^m c_i \lambda(x_i)$ and $\chi_H \cdot T_\alpha = \sum_{i=1}^n c_i \chi_H(x_i) \lambda(x_i)$ is a linear combination of the members of $\{\lambda_G(x) \colon x \in H\}$, we conclude that $T = \chi_H \cdot T = w^*\text{-}\lim_\alpha \chi_H \cdot T_\alpha \in \mathrm{VN}_H(G)$.

We will denote the inverse of the mapping $T \to \mathring{T}$ by Φ. For convenience, we will sometimes write $\Phi(T)$ as T'.

Lemma 4.2. *Let H be an open subgroup of a locally compact group G.*

(1) *For $T \in \mathrm{VN}_H(G)$ and $u \in A(H)$, $\langle \Phi(T), u \rangle = \langle T, \mathring{u} \rangle$.*
(2) *For $T \in \mathrm{VN}_H(G)$ and $v \in A(G)$, $\Phi(v \cdot T) = (v|H) \cdot \Phi(T)$.*
(3) *Let $T \in \mathrm{VN}_H(G)$. Then $T \in \mathrm{AP}(\widehat{G})$ if and only if $\Phi(T) \in \mathrm{AP}(\widehat{H})$.*

Proof. (1) Let $T \in \mathrm{VN}_H(G)$ and $u \in A(H)$. Then

$$\langle \Phi(T), u \rangle = \langle \Phi(T), \mathring{u}|H \rangle = \langle \Phi(T)^\circ, \mathring{u} \rangle = \langle T, \mathring{u} \rangle .$$

(2) Let $T \in \mathrm{VN}_H(G)$, $v \in A(G)$ and $u \in A(H)$. Then

$$\begin{aligned} \langle (v|H) \cdot \Phi(T), u \rangle &= \langle \Phi(T), (v|H)u \rangle \\ &= \langle T, ((v|H)u)^\circ \rangle, \quad \text{by (1)}, \\ &= \langle T, v\mathring{u} \rangle = \langle v \cdot T, \mathring{u} \rangle = \langle \Phi(v \cdot T), u \rangle . \end{aligned}$$

Therefore, $(v|H) \cdot \Phi(T) = \Phi(v \cdot T)$.

(3) Let $T \in \mathrm{VN}_H(G) \cap \mathrm{AP}(\widehat{G})$ and (u_n) a bounded sequence in $A(H)$. The $(\mathring{u}_n)$ is a bounded sequence in $A(G)$. Since $T \in \mathrm{AP}(\widehat{G})$, the sequence $\mathring{u}_n \cdot T$ has a norm convergent subsequence $\mathring{u}_{n_k} \cdot T$. Since Φ is an isometry, $\Phi(\mathring{u}_{n_k} \cdot T) = u_{n_k} \cdot \Phi(T)$ is also convergent in norm. Therefore, $\Phi(T) \in \mathrm{AP}(\widehat{H})$. Conversely, if $\Phi(T) \in \mathrm{AP}(\widehat{H})$ and v_n is a bounded sequence in $A(G)$ then the sequence $(v_n|H) \cdot \Phi(T)$ has a convergent subsequence $(v_{n_k}|H) \cdot \Phi(T)$. Since, by (2), $(v_n|H)\Phi(T) = \Phi(v_n \cdot T)$ and Φ is an isometry, we conclude that $v_{n_k} \cdot T$ converges in norm. Therefore $T \in \mathrm{AP}(\widehat{G})$.

Remark. Since both Φ and its inverse are weak-weak continuous, by applying the same proof we can also conclude the following. Let $T \in \mathrm{VN}_H(G)$. Then $T \in \mathrm{WAP}(\widehat{G})$ if and only if $\Phi(T) \in \mathrm{WAP}(\widehat{H})$.

Lemma 4.3. *Let H be an open normal subgroup of a locally compact group G such that G/H is a (discrete) amenable group. Then* $\mathrm{AP}(\widehat{G}) =$ *the norm closed linear span of* $\{\lambda_G(x)S\colon x \in G,\ S \in \mathrm{AP}(\widehat{G}) \cap \mathrm{VN}_H(G)\}$.

Proof. Let θ be the natural homomorphism of G onto G/H. Since G/H is an amenable discrete group, there exists an approximate identity (p_α) of $A(G/H)$ with $p_\alpha \in P_1(G/H) \cap A(G/H)$ (cf. Lau [30, Lemma 7.2]). It is clear that we may assume that the support of each p_α is finite. Let $q_\alpha = p_\alpha \circ \theta$. Then $q_\alpha \in P_1(G)$ (see [17, p. 199]) and $q_\alpha \to 1_G$ uniformly on compact subsets of G. By Theorem 2.4, $\|q_\alpha \cdot v - v\|_{A(G)} \to 0$ for each $v \in A(G)$. Therefore, for each $T \in \mathrm{VN}(G)$, $q_\alpha \cdot T - T$ converges to 0 in the w^*-topology: for $v \in A(G)$,

$$|\langle q_\alpha \cdot T - T, v\rangle| = \langle T, q_\alpha v - v\rangle| \le \|T\|\, \|q_\alpha v - v\|_{A(G)} \to 0.$$

Assume that $T \in \mathrm{AP}(\widehat{G})$. Then a subnet of $q_\alpha \cdot T$ converges in norm and, by the above paragraph, the limit is T. Let $\varepsilon > 0$ be given. Then there exists α_0 such that

$$(*) \qquad \|T - q_{\alpha_0} \cdot T\| < \varepsilon.$$

Since p_{α_0} is of finite support, $q_{\alpha_0} = \sum_{i=1}^n c_i \chi_{a_i H}$ for some $a_i \in G$ and $c_i \in \mathbf{C}$, and hence $q_{\alpha_0} \cdot T = \sum_{i=1}^n c_i(\chi_{a_i H} \cdot T)$. Since $\mathrm{AP}(\widehat{G})$ is a $B(G)$-module, $c_i \chi_{a_i H} \cdot T$ is a.p. and by Lemma 2.2(2),

$$\operatorname{supp}(c_i \chi_{a_i H} \cdot T) \subset (\operatorname{supp} \chi_{a_i H}) \cap \operatorname{supp} T \subset a_i H.$$

Let $T_i = \lambda(a_i^{-1})(c_i \chi_{a_i H} \cdot T)$. Then, by Lemma 2.2(2) $\operatorname{supp} T_i \subset a_i^{-1} \cdot (a_i H) = H$ and, by Lemma 2.1(2), $T_i \in \mathrm{AP}(\widehat{G})$. Thus

$$q_{\alpha_0} \cdot T = \sum_{i=1}^n \lambda(a_i) T_i$$

where $T_i \in \mathrm{AP}(\widehat{G})$ and $T_i \in \mathrm{VN}_H(G)$ (see Lemma 4.1). Since $\varepsilon > 0$ is arbitrary, together with $(*)$, we conclude that T belongs to the closed linear span of $\{\lambda(x)S\colon x \in G,\ S \in \mathrm{AP}(\widehat{G}) \cap \mathrm{VN}_H(G)\}$ and the proof is complete.

Theorem 4.4. *Let H be an open normal subgroup of a locally compact group G. Suppose that H has the dual Bohr approximation property and G/H is an amenable group. Then G also has the dual Bohr approximation property.*

Proof. Let $T \in \mathrm{AP}(\widehat{G})$. Let $\varepsilon > 0$ be given. By Lemma 4.3, there exist $a_i \in G$ and $T_i \in \mathrm{AP}(\widehat{G}) \cap \mathrm{VN}_H(G)$, $i = 1, 2, \ldots, n$, such that

$$(*) \qquad \left\| T - \sum_{i=1}^{n} \lambda(a_i) T_i \right\| < \varepsilon .$$

By Lemma 4.2(3), for each i, $\Phi(T_i) = T_i' \in \mathrm{AP}(\widehat{H})$. Since, by assumption, H has the dual Bohr approximation property, for each i, there exists

$$S_i' = \sum_{j=1}^{m_i} c_{i,j} \lambda_H(x_{i,j}) \in \mathrm{Trig}\, \widehat{H}$$

such that $\|T_i' - S_i'\|_{\mathrm{VN}(H)} < \varepsilon/n$. Let $S_i = \sum_{j=1}^{m_i} c_{i,j} \lambda_G(x_{i,j})$. Then $\Phi(S_i) = S_i'$. Since Φ is an isometry, $\|T_i - S_i\|_{\mathrm{VN}(G)} < \varepsilon/n$. Let $S = \sum_{i=1}^{n} \lambda(a_i) S_i$. Then $S \in \mathrm{Trig}\, \widehat{G}$ and

$$(**) \qquad \left\| S - \sum_{i=1}^{n} \lambda(a_i) T_i \right\| \le \sum_{i=1}^{n} \|\lambda(a_i)(S_i - T_i)\| \le \sum_{i=1}^{n} \|S_i - T_i\| < \varepsilon .$$

Thus, by $(*)$ and $(**)$,

$$\|T - S\| \le \left\| T - \sum_{i=1}^{n} \lambda(a_i) T_i \right\| + \left\| \sum_{i=1}^{n} \lambda(a_i) T_i - S \right\| < 2\varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, $T \in C_\delta^*(G)$. Thus $\mathrm{AP}(\widehat{G}) = C_\delta^*(G)$.

Since locally compact abelian groups have the dual Bohr approximation property, we have the following.

Corollary 4.5. *If G is an extension of a locally compact abelian group by a discrete amenable group then G has the dual Bohr approximation property.*

Remarks. (1) If a locally compact group G is of bounded representation type, i.e., $\sup\{d_\sigma : \sigma \in \widehat{G}\} < \infty$, then by a theorem of C. C. Moore [36], G is a finite extension of a locally compact abelian group and hence G has the dual Bohr approximation property; in particular, $\mathrm{AP}(\widehat{G})$ is a C^*-algebra. Dunkl and Ramirez have proved in [14] that $\mathrm{AP}(\widehat{G})$ is a C^*-algebra if G is a compact group of bounded representation type.

(2) We would like to mention three interesting examples of groups that are extensions of compact abelian groups by discrete abelian groups, all taken from the survey article of Palmer [39].

(i) Let $G_4 = A \times B$ (semidirect product) where $A = \sum_{n=1}^{\infty} \mathbf{Z}_2$, the weak sum of countably many copies of $\mathbf{Z}_2 = \{1, -1\}$, and $B = \prod_{n=1}^{\infty} \mathbf{T}$, the product of countably many copies of $\mathbf{T}$, and the action of A on B is given by $(a \cdot b)(n) = (b(n))^{a(n)}$.

(ii) Let $G_5 = \mathbf{Z}^2 \times \mathbf{T}$ with multiplication $(m_1, m_2, t)(n_1, n_2, s) = (m_1 + n_1, m_2 + n_2, tse^{im_1n_2})$.

(iii) Let $G_6 = \mathbf{Z} \times K$ (semidirect product) where $K = \prod_{n=-\infty}^{\infty} \mathbf{Z}_2$ and the action of $\mathbf{Z}$ on K is given by

$$(n \cdot k)(m) = k(m - n), \qquad k \in K, \ m, n \in \mathbf{Z}.$$

The group G_4 was first considered by C. C. Moore [36]. He showed that it is a Moore group, i.e., $d_\sigma < \infty$ for each $\sigma \in G$, but it is not a finite extension of a Z-group; in particular, G_4 is not of bounded representation type, G_5 is a central extension of $\mathbf{T}$ by $\mathbf{Z}^2$ and hence is nilpotent, but it is not of type I. G_6 is a CCR-group but it is not a maximal almost periodic group and hence is not a Moore group. By Corollary 4.5, G_4, G_5 and G_6 have the dual Bohr approximation property.

Dunkl and Ramirez proved in [14] that if a compact group G is of bounded representation type then both AP($\widehat{G}$) and WAP($\widehat{G}$) are C^*-algebras. We have the following generalization of their result.

Proposition 4.6. *Let H be an open normal subgroup of a locally compact group G such that G/H is amenable.*

(1) *If* AP($\widehat{H}$) *is a C^*-algebra then so is* AP($\widehat{G}$).

(2) *If* WAP($\widehat{H}$) *is a C^*-algebra then so is* WAP($\widehat{G}$).

Proof. (1) Let $S, T \in$ AP($\widehat{G}$). Let $\varepsilon > 0$ be given. By Lemma 4.3, there exist $V = \sum_{i=1}^{n} \lambda(a_i)S_i$ and $W = \sum_{j=1}^{m} \lambda(b_j)T_j$ where $S_i, T_j \in$ AP($\widehat{G}$) $\cap$ VN$_H(G)$, $a_i, b_j \in G$, such that $\|S - V\| < \varepsilon$, $\|T - W\| < \varepsilon$. So

$$(*) \qquad \|ST - VW\| \le \varepsilon(\|S\| + \|T\| + \varepsilon).$$

Let $A_{i,j} = \lambda(b_j^{-1})S_i\lambda(b_j)$. Then, by Lemma 2.1(2), Lemma 2.2(2) and Lemma 4.1, $A_{i,j} \in$ AP($\widehat{G}$) $\cap$ VN$_H(G)$. Therefore, by Lemma 4.2(3), $A'_{i,j}$ and T'_j belong to AP($\widehat{H}$). By assumption, $A'_{ij}T'_j \in$ AP($\widehat{H}$). Since $(A_{ij}T_j)' = A'_{ij}T'_j$, by Lemma 4.2(3), $A_{ij}T_j \in$ AP($\widehat{G}$) $\cap$ VN$_H(G)$. Therefore, by Lemma 2.1(2) again,

$$\begin{aligned} VW &= \sum_{i=1}^{n}\sum_{j=1}^{m} \lambda(a_i)S_i\lambda(b_j)T_j \\ &= \sum_{i,j} \lambda(a_i)\lambda(b_j)(\lambda(b_j^{-1})S_i\lambda(b_j))T_j \\ &= \sum_{i,j} \lambda(a_ib_j)A_{ij}T_j \end{aligned}$$

belongs to AP($\widehat{G}$). Since AP($\widehat{G}$) is norm closed and $\varepsilon > 0$ is arbitrary, by $(*)$, $ST \in$ AP($\widehat{G}$). Thus AP($\widehat{G}$) is closed under multiplication and hence is a C^*-algebra.

(2) Using the same proof as that of Lemma 4.3, we can show that if $T \in \mathrm{WAP}(\widehat{G})$ then T is contained in the weak closed linear span of $Y = \{\lambda(x)S: x \in G,\ S \in \mathrm{WAP}(\widehat{G}) \cap \mathrm{VN}_H(G)\}$. Therefore, T also belongs to the norm closed linear span of Y. Now applying the same proof as that of (1), using the remark after Lemma 4.2, we can conclude that $\mathrm{WAP}(\widehat{G})$ is an algebra.

Remark. It is not known, in general, whether $\mathrm{AP}(\widehat{G})$ is an algebra for a general locally compact group G.

5. A stronger almost periodicity

As mentioned earlier, Dunkl and Ramirez have shown in [16] that, for an abelian locally compact group G, $f \in L^\infty(G)$ is a.p. (w.a.p.) if and only if $g \mapsto g * f$ is a compact (weakly compact) operator from $L^1(G)$ into $L^\infty(G)$. Their proof also works for nonabelian groups. In the following lemma we will add an additional equivalent condition. For completeness we will outline the proofs of all the implications.

Lemma 5.1. *Let G be a locally compact group, $f \in L^\infty(G)$. Then the following three statements are equivalent.*

(1) $f \in \mathrm{AP}(G)$.

(2) $g \mapsto g * f$ *is a compact operator from* $L^1(G)$ *into* $L^\infty(G)$.

(3) $\nu \mapsto \nu * f$ *is a compact operator from* $M(G)$ *into* $L^\infty(G)$.

(*Here $M(G)$ is the algebra of bounded regular Borel measures on G with the total variation norm.*)

Proof. (1) $\Rightarrow$ (2) Let $P = \{f \in L^1(G): f \geq 0,\ \|f\|_1 = 1\}$. If $f \in \mathrm{AP}(G)$, then $\{{}_tf: t \in G\}$ is relatively compact. Thus the norm closed convex hull $K(f)$ of $\{{}_tf: t \in G\}$ is compact in $L^\infty(G)$. To see that $g \mapsto g * f$ is compact from $L^1(G)$ to $L^\infty(G)$, one only has to note that since f is uniformly continuous, $\{g * f: g \in P\} \subset K_f$.

(2) $\Rightarrow$ (3) By (2), $P * f = \{g * f: g \in P\}$ is relatively compact. Let $K = \{\nu \in M(G): \|\nu\| = 1, \nu \geq 0\}$. To show that $\nu \mapsto \nu * f$ is compact on $M(G)$, it suffices to show that $\{\nu * f: \nu \in K\}$ is contained in $(P * f)^-$, the norm closure of $P * f$. Let (g_α) be a left approximate identity of $L^1(G)$ with $g_\alpha \in P$, i.e., $g_\alpha * g \to g$ in L^1-norm for each $g \in L^1(G)$. Then $g_\alpha * f - f \to 0$ in $\sigma(L^\infty(G), L^1(G))$. Since $P * f$ is relatively compact, there is a subnet $(g_{\alpha'})$ of (g_α) such that $g_{\alpha'} * f \to f$ in norm. Let $\nu \in K$. Then $(\nu * g_{\alpha'}) * f \to \nu * f$ in norm and $\nu * g_{\alpha'} \in P$. Thus $\nu * f \in (P * f)^-$, as wanted.

(3) $\Rightarrow$ (1) is obvious, since $\{{}_tf: t \in G\} \subset K * f$ and by (3), $K * f$ is relatively compact.

Remark. A similar proof will show that for $f \in L^\infty(G)$, $f \in \mathrm{WAP}(G) \Leftrightarrow \nu \mapsto \nu * f$ is a weakly compact operator from $M(G)$ into $L^\infty(G)$.

When G is abelian, $B_\lambda(G) = B(G) \cong M(\widehat{G})$ and the module action of $B(G)$ on $\mathrm{VN}(G)$ corresponds to the convolution of $\nu \in M(\widehat{G})$ and $f \in L^\infty(\widehat{G})$.

Thus $f \in L^\infty(\widehat{G})$ is a.p. if and only if the corresponding operator $T \in \mathrm{VN}(G)$ satisfies the following: $u \mapsto u \cdot T$ is a compact operator from $B_\lambda(G) = B(G)$ into $\mathrm{VN}(G)$. For a general locally compact group G and $T \in \mathrm{VN}(G)$ we will denote the operator $u \mapsto u \cdot T$ from $B_\lambda(G)$ into $\mathrm{VN}(G)$ by T^{B_λ} and the operator $u \mapsto u \cdot T$ from $B(G)$ into $\mathrm{VN}(G)$ by T^B. $T \in \mathrm{VN}(G)$ is said to be $B_\lambda(G)$-almost periodic (B_λ-a.p.) if T^{B_λ} is a compact operator. Similarly, T is $B(G)$-almost periodic (B-a.p.) if T^B is a compact operator. Both $B_\lambda(G)$ and $B(G)$ almost periodicities are natural generalizations of almost periodicity of bounded functions in the dual of an abelian group. It is obvious that T is B-a.p. $\Rightarrow$ T is B_λ-a.p. $\Rightarrow$ T is a.p. We will show that if T is a.p. then T is B_λ-a.p. and hence for amenable G, if T is a.p. then T is B-a.p. Let $\mathrm{AP}_B(\widehat{G}) = \{T \in \mathrm{VN}(G) : T \text{ is } B\text{-a.p.}\}$. Similarly, $T \in \mathrm{VN}(G)$ is said to be $B(G)$-weakly almost periodic (B-w.a.p.) if T^B is a weakly compact operator. Let $\mathrm{WAP}_B(\widehat{G}) = \{T \in \mathrm{VN}(G) : T \text{ is } B\text{-w.a.p.}\}$.

Proposition 5.2. *Let G be a locally compact group, $T \in \mathrm{VN}(G)$. Then the following two conditions are equivalent:*

(1) *T is a.p.*

(2) *T is B_λ-a.p.*

Proof. (2) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (2) Assume that T is a.p. Then the set

$$D = \{u \cdot T : u \in A(G), \|u\|_{A(G)} \le 2\}$$

is relatively compact in $\mathrm{VN}(G)$. To show that T^{B_λ} is a compact operator, it suffices to show that the set

$$\{u \cdot T : u \in B_\lambda(G), \|u\|_{B_\lambda(G)} \le 1\}$$

is contained in the norm closure of D. Let $u \in B_\lambda(G)$, $\|u\|_{B(G)} \le 1$, be given. Since $B_\lambda(G)$ is the dual of the C^*-algebra $C^*_\lambda(G)$, u can be written as $u = u' - iu''$ where u', u'' are hermitian elements of $B_\lambda(G)$ and $\|u'\|_{B(G)} \le 1$, $\|u''\|_{B(G)} \le 1$ (see Dixmier [10, p. 6]). Let $u' = u_1 - u_2$, $u'' = u_3 - u_4$ be the orthogonal decomposition of u' and u'' where $u_j \in B_\lambda(G) \cap P(G)$ and $\|u_1\| + \|u_2\| = \|u'\|$, $\|u_3\| + \|u_4\| = \|u''\|$ (see [10, p. 273]). Since each u_j belongs to $B_\lambda(G) \cap P(G)$, there exist nets $u_{j,\alpha}$ in $P(G) \cap A(G)$, indexed by the same directed set $\{\alpha\}$, such that $\|u_{j,\alpha}\|_{B(G)} \le \|u_j\|_{B(G)}$ and $u_{j,\alpha} \to u_j$ uniformly on compact subsets of G (see [10, p. 357]). By Theorem 2.4,

$$\lim_\alpha \|u_{j,\alpha} v - u_j v\|_{A(G)} = 0$$

for $v \in A(G)$ and $j = 1, 2, 3, 4$. Let $u_\alpha = (u_{1,\alpha} - u_{2,\alpha}) + i(u_{3,\alpha} - u_{4,\alpha})$. Then $u_\alpha \in A(G)$, $\|u_\alpha\|_{A(G)} \le 2$ and $\lim_\alpha \|u_\alpha v - uv\|_{A(G)} = 0$. Therefore, $u_\alpha \cdot T \to u \cdot T$ in the w^*-topology. Since $T \in \mathrm{AP}(\widehat{G})$, a subnet of $(u_\alpha \cdot T)$

converges in norm to $u \cdot T$. Thus $u \cdot T$ belongs to the norm closure of D, as wanted.

Remark. By exactly the same proof, we can show that, for $T \in \mathrm{VN}(G)$, $T \in \mathrm{WAP}(\widehat{G})$ if and only if T^{B_λ} is weakly compact.

Proposition 5.3. *Let G be an amenable locally compact group and $T \in \mathrm{VN}(G)$. Then T is almost periodic if and only if T is $B(G)$-almost periodic.*

Proof. Since G is amenable, $B(G) = B_\lambda(G)$. The result follows directly from the above proposition.

Remark. The above proposition can also be proved directly by following the same arguments as the proof of (2) $\Rightarrow$ (3) of Proposition 5.1, using the existence of bounded approximate identity for $A(G)$.

As mentioned earlier, if G is an amenable discrete group then $\mathrm{WAP}(\widehat{G}) = \mathrm{AP}(\widehat{G}) = C^*_\delta(G)$ (see [20]) and hence $\mathrm{WAP}_B(\widehat{G}) = C^*_\delta(G)$. On the other hand, we do not know whether $\mathrm{AP}(\widehat{F}_r) = C^*_\delta(\mathbf{F}_r)$ where $\mathbf{F}_r$ is the free group on r generators, $1 < r < \infty$; but we are able to show that $\mathrm{WAP}_B(\widehat{F}_r) = C^*_\delta(\mathbf{F}_r)$.

If G is a discrete group and $T \in \mathrm{VN}(G)$ then $\varphi = T(\delta_e) \in l^2(G)$ where $\delta_e = \chi_{\{e\}}$, and for each $h \in l^2(G)$, $T(h) = \varphi * h$. Therefore as is usually done, we will identify T with $\varphi \in l^2(G)$. Thus if $\varphi \in \mathrm{VN}(G) \subset l^2(G)$, then as an operator on $l^2(G)$, φ sends $h \in l^2(G)$ to $\varphi * h$ (see [18]). With this identification, the module action of $B(G)$ on $\mathrm{VN}(G)$ is just the pointwise multiplication of functions.

Theorem 5.4. *Let $G = \mathbf{F}_r$, $1 < r < \infty$. Then* $\mathrm{WAP}_B(\widehat{G}) = \mathrm{AP}_B(\widehat{G}) = C^*_\delta(G)$.

Proof. If $x \in G = \mathbf{F}_r$, $|x|$ denotes the length of the reduced word for x. Let $X_n = \{x \in G\colon |x| = n\}$ and χ_n the characteristic function of X_n. For $t > 0$, let u_t be the function on G defined by $u_t(x) = e^{-t|x|}$. Haagerup proved in [24, Lemma 1.2] that $u_t \in P_1(G)$. We claim that for $t > 0$ and $\varphi \in \mathrm{VN}(G)$, $u_t \cdot \varphi \in C^*_\lambda(G) = C^*_\delta(G)$. By Lemma 1.4 of Haagerup [24] (see also Figă-Talamanca and Picardello [18, p. 13]) it suffices to show that

$$\sum_{n=0}^{\infty} \|u_t \cdot \varphi \cdot \chi_n\|_2 (n+1) < \infty .$$

Indeed,

$$\|u_t \cdot \varphi \cdot \chi_n\|_2^2 = \sum_{x \in X_n} |u_t(x)\varphi(x)|^2 \le e^{-2tn} \sum_{x \in X_n} |\varphi(x)|^2 \le e^{-2tn} \|\varphi\|_2^2 ,$$

and hence

$$\sum_{n=0}^{\infty} \|u_t \cdot \varphi \cdot \chi_n\|_2 (n+1) \le \left(\sum_{n=0}^{\infty} e^{-tn}(n+1) \right) \|\varphi\|_2 < \infty .$$

Choose positive numbers t_n such that $\lim_n t_n = 0$. Let $p_n = u_{t_n}$. Then
(i) $p_n \in P_1(G)$ and $\lim_n p_n = 1_G$ pointwise, and
(ii) $p_n \cdot \mathrm{VN}(G) \subset C_\delta^*(G)$.
By Theorem 2.4, $\lim_n \|p_n v - v\|_{A(G)} = 0$. Therefore, for any $\varphi \in \mathrm{VN}(G)$, $w^*\text{-}\lim_n(p_n \cdot \varphi - \varphi) = 0$. In particular, if $\varphi \in \mathrm{WAP}_B(\widehat{G})$ then a subsequence of $p_n \cdot \varphi$ converges to φ in w-topology. Since $p_n \cdot \varphi \in C_\delta^*(G)$ for each n and $C_\delta^*(G)$ is weakly closed, we conclude that $\varphi \in C_\delta^*(G)$. Thus $\mathrm{WAP}_B(\widehat{G}) = C_\delta^*(G)$.

Granirer has pointed out in [20] that if G is an amenable locally compact group then $\mathrm{AP}(\widehat{G}) \subset \mathrm{WAP}(\widehat{G}) \subset \mathrm{UC}(\widehat{G})$. It is still unknown whether $\mathrm{AP}(\widehat{G})$ is contained in $\mathrm{UC}(\widehat{G})$ when G is not amenable. But we will show that there exist nonamenable nondiscrete groups G such that $\mathrm{WAP}_B(\widehat{G}) \subset \mathrm{UC}(\widehat{G})$.

Definition. Let G be a locally compact group. A net of functions (p_α) in $P_1(G)$ is called a $(*)$-net if
(i) $\lim_\alpha p_\alpha = 1_G$ uniformly on compact subsets of G, and
(ii) for each α, $p_\alpha \cdot \mathrm{VN}(G) \subset \mathrm{UC}(\widehat{G})$.

Condition (ii) above says that p_α is a $\mathrm{VN}(G) - \mathrm{UC}(\widehat{G})$ multiplier. In the proof of Theorem 5.4, we have shown that $\mathbf{F}_r$ has a $(*)$-sequence. Also note that a $(*)$-net exists for any amenable group G: just take an approximate identity (p_α) of $A(G)$ with $p_\alpha \in A(G) \cap P_1(G)$. Note that if (p_α) is a $(*)$-net for a nonamenable G then there exists α_0 such that if $\alpha \geq \alpha_0$ then $p_\alpha \notin B_\lambda(G)$; in particular, $p_\alpha \notin A(G)$ but $p_\alpha \cdot \mathrm{VN}(G) \subset \mathrm{UC}(\widehat{G})$. On the other hand, it is not hard to show that, for an amenable G, if $u \in B(G)$ and $u \cdot \mathrm{VN}(G) \subset \mathrm{UC}(\widehat{G})$ then $u \in A(G)$.

By Theorem 2.4, we can conclude as before that if G has a $(*)$-net (p_α) then $w^*\text{-}\lim_\alpha p_\alpha \cdot T = T$ for each $T \in \mathrm{VN}(G)$. Thus, as in the proof of Theorem 5.4, we have the following.

Proposition 5.5. *Let G be a locally compact group with a $(*)$-net (p_α). Then* $\mathrm{WAP}_B(\widehat{G}) \subset \mathrm{UC}(\widehat{G})$. *In particular, if G is, in addition, discrete, then* $\mathrm{WAP}_B(G) = \mathrm{AP}_B(\widehat{G}) = C_\delta^*(G)$.

For a locally compact group G, let $MA(G) = \{u\colon u$ a bounded continuous function on G and $u \cdot A(G) \subset A(G)\}$, the space of multipliers of $A(G)$, with the multiplier norm: $\|u\|_{MA(G)} = \sup\{\|u \cdot v\|_{A(G)}\colon v \in A(G), \|v\|_{A(G)} \leq 1\}$. It is known that if G is amenable then $MA(G) = B(G)$ (see Pier [40, p. 209]). Losert [31] has shown that the converse is also true: $MA(G) = B(G) \Rightarrow G$ is amenable (see also Nebbia [38], for the discrete case). Cowling [8] has proved that the coefficient functions of (strongly continuous) uniformly bounded representations of G are multipliers of $A(G)$.

For a positive integer n, $n \geq 2$, $\mathrm{SO}(n, 1)$ is the group of real $(n+1)\times(n+1)$ matrices with determinant 1 that leave the quadratic form $-t_0^2 + t_1^2 + \cdots + t_n^2$

invariant and $\mathrm{SO}_0(n,1)$ is the connected component of $\mathrm{SO}(n,1)$. Let K be the maximal compact subgroup of $G = \mathrm{SO}_0(n,1)$ consisting of matrices of the form $\left(\begin{smallmatrix}1 & 0\\ 0 & k\end{smallmatrix}\right)$ where $k \in \mathrm{SO}(n)$. In [4] de Canniere and Haagerup gave a nice summary of some known results on the spherical functions φ_s, $s \in \mathbf{C}$, of G with respect to K. They also obtained several interesting new results on these functions. Here are some of the results that are relevant to us:

(1) $\varphi_s \in P_1(G)$, if either $\mathrm{re}\, s = 0$ or s is real and $|s| \le (n-1)/2$; $\varphi_s = 1_G$ if $s = \pm(n-1)/2$.

(2) The mapping $s \to \varphi_s$ is continuous on $\mathbf{C}$ when the space of continous functions on G is equipped with the topology of uniform convergence on compact subsets.

(3) If $|\mathrm{re}\, s| < (n-1)/2$, then φ_s is a coefficient function of a uniformly bounded representation of G and hence $\varphi_s \in \mathrm{MA}(G)$. Furthermore, $s \to \varphi_s$ is continuous from $\{s \in \mathbf{C}\colon |\mathrm{re}\, s| < (n-1)/2\}$ to $\mathrm{MA}(G)$ and $\|\varphi_{\sigma+it}\|_{\mathrm{MA}(G)} \le C_\sigma(1+4|t|)^{3(n-1)/2}$ where $t, \sigma \in \mathbf{R}$, $|\sigma| < (n-1)/2$ and C_σ is a finite constant, depending on σ (see [4, Proposition 3.5]).

Theorem 5.6. *The group* $G = \mathrm{SO}_0(n,1)$, $n \ge 2$, *has a* $(*)$*-sequence.*

Proof. Choose $\sigma_k \in \mathbf{R}$ such that $0 < \sigma_k < (n-1)/2$ and $\lim_k \sigma_k = (n-1)/2$. Let $p_k = \varphi_{\sigma_k}$. Then, by (1) and (2) above, $p_k \in P_1(G)$ and $p_k \to 1_G$ uniformly on compact subsets of G. To show that (p_k) is a $(*)$-sequence, it remains to show that $\varphi_\sigma \cdot \mathrm{VN}(G) \subset \mathrm{UC}(\widehat{G})$ for each σ, $0 < \sigma < (n-1)/2$. As in the proof of Theorem 3.7 of [4], let

$$\varphi_{\sigma,m} = \frac{m}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-m^2 t^2)\varphi_{\sigma+it}\, dt, \qquad m = 1, 2, \dots .$$

The integral is a strong Banach space integral in $\mathrm{MA}(G)$; it exists, because of (3) stated above. Again, by the proof of Theorem 3.7 of [4], $\varphi_{\sigma,m} \in A(G)$ and $\lim_m \|\varphi_{\sigma,m} - \varphi_\sigma\|_{\mathrm{MA}(G)} = 0$. So, for $T \in \mathrm{VN}(G)$,

$$\begin{aligned}\|\varphi_{\sigma,m}T - \varphi_\sigma \cdot T\|_{\mathrm{VN}(G)} &= \sup\{|\langle \varphi_{\sigma,m}T - \varphi_\sigma \cdot T, u\rangle| \colon u \in A(G), \|u\|_{A(G)} \le 1\}\\ &= \sup\{|\langle T, \varphi_{\sigma,m}u - \varphi_\sigma u\rangle| \colon u \in A(G), \|u\|_{A(G)} \le 1\}\\ &\le \sup\{\|T\|\, \|\varphi_{\sigma,m}u - \varphi_\sigma u\|_{A(G)} \colon u \in A(G), \|u\|_{A(G)} \le 1\}\\ &\le \|T\|\, \|\varphi_{\sigma,m} - \varphi_\sigma\|_{\mathrm{MA}(G)} \to 0, \quad \text{as } m \to \infty.\end{aligned}$$

By definition, $\varphi_{\sigma,m} \cdot T \in A(G) \cdot \mathrm{VN}(G) \subset \mathrm{UC}(\widehat{G})$, and hence $\varphi_\sigma \cdot T \in \mathrm{UC}(\widehat{G})$. The proof is complete.

Remark. In [4], de Canniere and Haagerup considered $\mathrm{MA}_0(G) =$ the space of completely bounded multipliers of $A(G)$, instead of $\mathrm{MA}(G)$. But their results also hold in the $\mathrm{MA}(G)$ setting, since $\mathrm{MA}_0(G) \subset \mathrm{MA}(G)$ and for $u \in \mathrm{MA}_0(G)$, $\|u\|_{\mathrm{MA}(G)} \le \|u\|_{\mathrm{MA}_0(G)}$.

Corollary 5.7. *If* H *is a closed subgroup of* $\mathrm{SO}_0(n,1)$ *then* H *has a* $(*)$*-sequence.*

Proof. Let $p_k = \varphi_{\sigma_k}$ be as in the above proof. Let $p'_k = p_k|H$. Then $p'_k \in P_1(H)$ and $p'_k \to 1_H$ uniformly on compact subsets of H. To see that (p'_k) is a $(*)$-sequence, we only have to show that $\varphi'_\sigma \cdot \mathrm{VN}(H) \subset \mathrm{UC}(\widehat{H})$ where $0 < \sigma < (n-1)/2$ and $\varphi'_\sigma = \varphi_\sigma|H$. Let $\varphi'_{\sigma,m} = \varphi_{\sigma,m}|H$. Then $\varphi'_{\sigma,m} \in A(H)$ and, by Proposition 1.12 of [4], $\|\varphi'_\sigma - \varphi'_{\sigma,m}\|_{\mathrm{MA}(H)} \le \|\varphi_\sigma - \varphi_{\sigma,m}\|_{\mathrm{MA}(G)} \to 0$, as $m \to \infty$. Thus, as before, $\varphi'_\sigma \cdot \mathrm{VN}(H) \subset \mathrm{UC}(\widehat{H})$ and the proof is complete.

Corollary 5.8. *Let H be a closed subgroup of $\mathrm{SO}_0(n,1)$. Then $\mathrm{WAP}_B(\widehat{H}) \subset \mathrm{UC}(\widehat{H})$. In particular, if H is discrete then $\mathrm{WAP}_B(\widehat{H}) = C^*_\delta(H)$.*

Remarks. (1) Since $\mathrm{SO}_0(2,1)$ contains $\mathbf{F}_r$ as a discrete subgroup, the above corollary also implies Theorem 5.4. But we prefer to give an independent proof for the case $\mathbf{F}_r$ by using the Haagerup functions $u_t(x) = e^{-t|x|}$. Note that Figà-Talamanka and Picardello have also constructed spherical functions φ_z, $z \in \mathbf{C}$, for $\mathbf{F}_r$ (see [18, p. 36]). Choose $\sigma_k \in \mathbf{R}$, $\frac{1}{2} < \sigma_k < 1$ and $\sigma_k \to 1$. Let $p_k = \varphi_{\sigma_k}$. Then (p_k) is also a $(*)$-sequence for $\mathbf{F}_r$.

(2) By applying the same arguments as in §4, we can show the following: if H is an open normal subgroup of a locally compact group G such that G/H is amenable and $\mathrm{WAP}_B(\widehat{H}) \subset \mathrm{UC}(\widehat{H})$ then $\mathrm{WAP}_B(\widehat{G}) \subset \mathrm{UC}(\widehat{G})$. If $G = \mathrm{SL}(2,\mathbf{Z})$ then it has a normal subgroup H such that H is of finite index in G and $H \cong \mathbf{F}_2$ (see [32, p. 100]) and hence $\mathrm{WAP}_B(\widehat{G}) = C^*_\delta(G)$.

We believe that the existence of $(*)$-net for a given group G should be useful in other aspects of analysis on G. But we would like to conclude this paper with the following.

Theorem 5.9. *Let G be a noncompact locally compact group with Kazhdan's property* (T). *Then G does not have a $(*)$-net.*

Proof. Suppose that G has property (T) and let (p_α) be a $(*)$-net. Then $p_\alpha \to 1_G$ uniformly on compact subsets. By Akemann-Walter [1, Lemma 2], $p_\alpha \to 1_G$ in $B(G)$-norm. In particular, for $T \in \mathrm{VN}(G)$,

$$\|p_\alpha \cdot T - T\| \le \|p_\alpha - 1_G\|_{B(G)}\|T\| \to 0.$$

Therefore, $T \in \mathrm{UC}(\widehat{G})$. Thus $\mathrm{VN}(G) = \mathrm{UC}(\widehat{G})$. By Granirer [21, Proposition 2; 22, Theorem 3], G is compact.

In particular, $\mathrm{SL}(3,\mathbf{Z})$ does not have a $(*)$-sequence. We are unable to decide whether $\mathrm{WAP}_B(\widehat{G}) = C^*_\delta(G)$ if $G = \mathrm{SL}(3,\mathbf{Z})$.

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